Digital Communication Systems
EES 452

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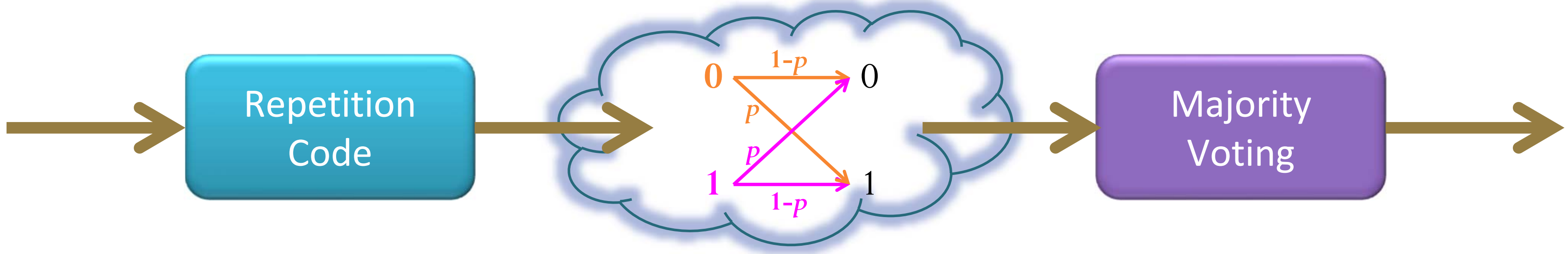
4. Mutual Information and Channel Capacity

4.2 Operational Channel Capacity

Example: Repetition Code

[Section 3.5]

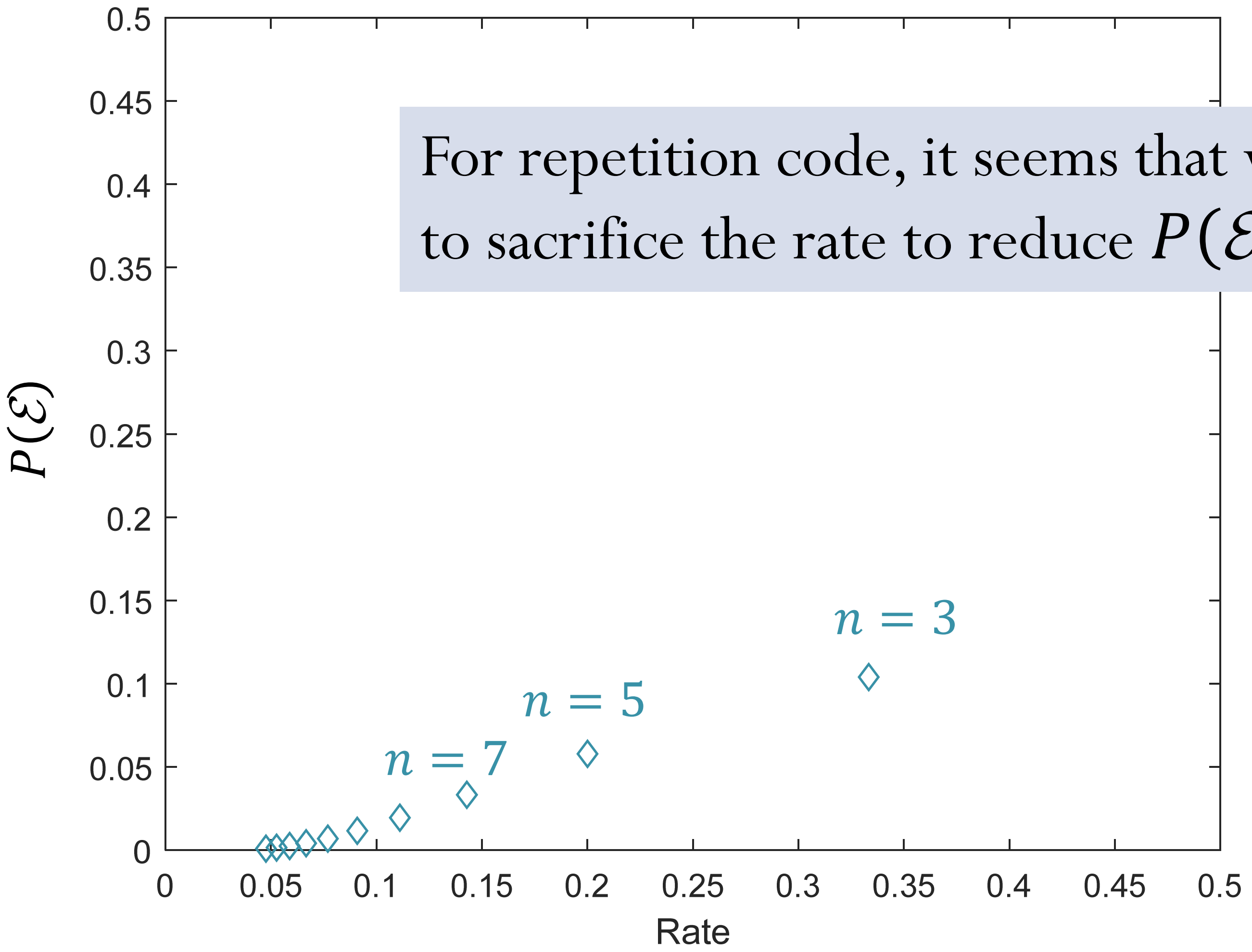
BSC with $p = 0.2$



n	$P(\mathcal{E})$ Probability that more than half of the bits are in error	Code Rate
1	$p = 0.2$	$\frac{1}{1} = 1$
3	$\binom{3}{2} p^2(1-p) + \binom{3}{3} p^3 \approx 0.1040$	$\frac{1}{3} \approx 0.33$
5	$\binom{5}{3} p^3(1-p)^2 + \binom{5}{4} p^4(1-p)^1 + \binom{5}{5} p^5 \approx 0.0579$	$\frac{1}{5} = 0.2$
7	≈ 0.0333	$\frac{1}{7} \approx 0.1429$
9	≈ 0.0196	$\frac{1}{9} \approx 0.1111$
11	≈ 0.0117	$\frac{1}{11} \approx 0.0909$

Achievable Performance

BSC with $p = 0.2$ Repetition Code ($k = 1$)



For repetition code, it seems that we have to sacrifice the rate to reduce $P(\mathcal{E})$

Designing Channel Encoder

2^k rows

$\underline{\mathbf{s}}$	$\underline{\mathbf{x}}$
00	? ? ? ? ?
01	? ? ? ? ?
10	? ? ? ? ?
11	? ? ? ? ?

n columns

Each "?" can be 0 or 1.

So, there are

$$2^{(n2^k)}$$ $= 1{,}048{,}576$ for $n = 5$, $k = 2$

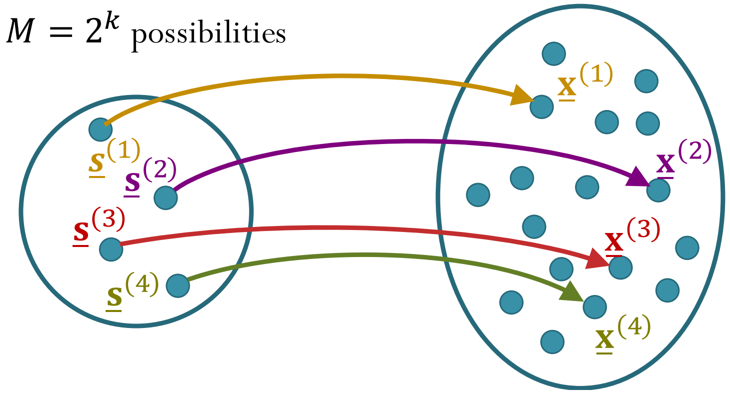
possibilities.

But we don't want to use the same codeword to represent two different info blocks.

So, actually, we need to consider

$$\binom{2^n}{2^k}$$ $= 35{,}960$ for $n = 5$, $k = 2$

possibilities.



Choose $M = 2^k$ from 2^n possibilities to be used as codewords.

MATLAB

```
close all; clear all;

% EES315 2020 Example 6.58
% EES452 2020 Examples 3.62, 3.67
C = [0 0 0 0 0; 1 1 1 1 1]; % repetition code

p = (1/100);
PE_minDist(C,p)
```

Code C is defined by putting all its (valid) codewords as its rows. For repetition code, there are two codewords: 00..0 and 11..1.

Crossover probability of the binary symmetric channel.

```
>> PE_minDist_demo1

ans =
   9.8506e-06
```

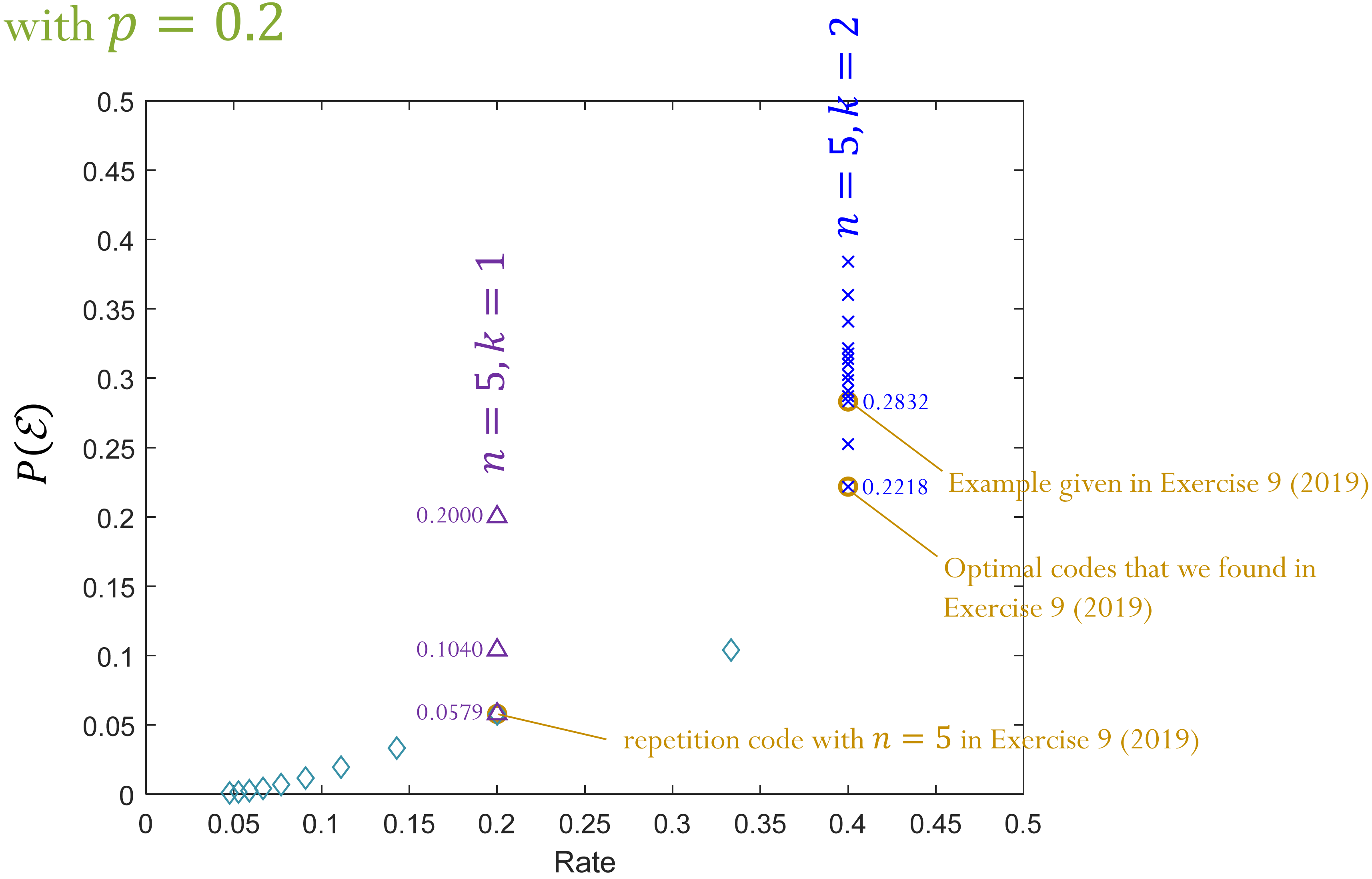
PE_minDist.m

MATLAB

```
function PE = PE_minDist(C,p)
% Function PE_minDist computes the error probability P(E) when code C
% is used for transmission over BSC with crossover probability p.
% Code C is defined by putting all its (valid) codewords as its rows.
M = size(C,1); % the number of (valid) codewords
k = log2(M);
n = size(C,2);

% Generate all possible n-bit received vectors
Y = dec2bin(0:2^n-1)-'0';

% Normally, we need to construct an extended Q matrix. However, because
% each conditional probability in there is a decreasing function of the
% (Hamming) distance, we can work with the distances instead of the
% conditional probability. In particular, instead of selecting the max in
% each column of the Q matrix, we consider min distance in each column.
dminy = zeros(1,2^n); % preallocation
for j = 1:(2^n)
    % for each received vector y,
    y = Y(j,:);
    % find the minimum distance
    % (the distance from y to the closest codeword)
    d = sum(mod(bsxfun(@plus,y,C),2),2);
    dminy(j) = min(d);
end

% From the distances, calculate the conditional probabilities.
% Note that we compute only the values that are to be selected (instead of
% calculating the whole Q first).
n1 = dminy; n0 = n-dminy;
Qmax = (p.^n1).*((1-p).^n0);
% Scale the conditional probabilities by the input probabilities and add
% the values. Note that we assume equally likely input.
PC = sum((1/M)*Qmax);
PE = 1-PC;
end
```

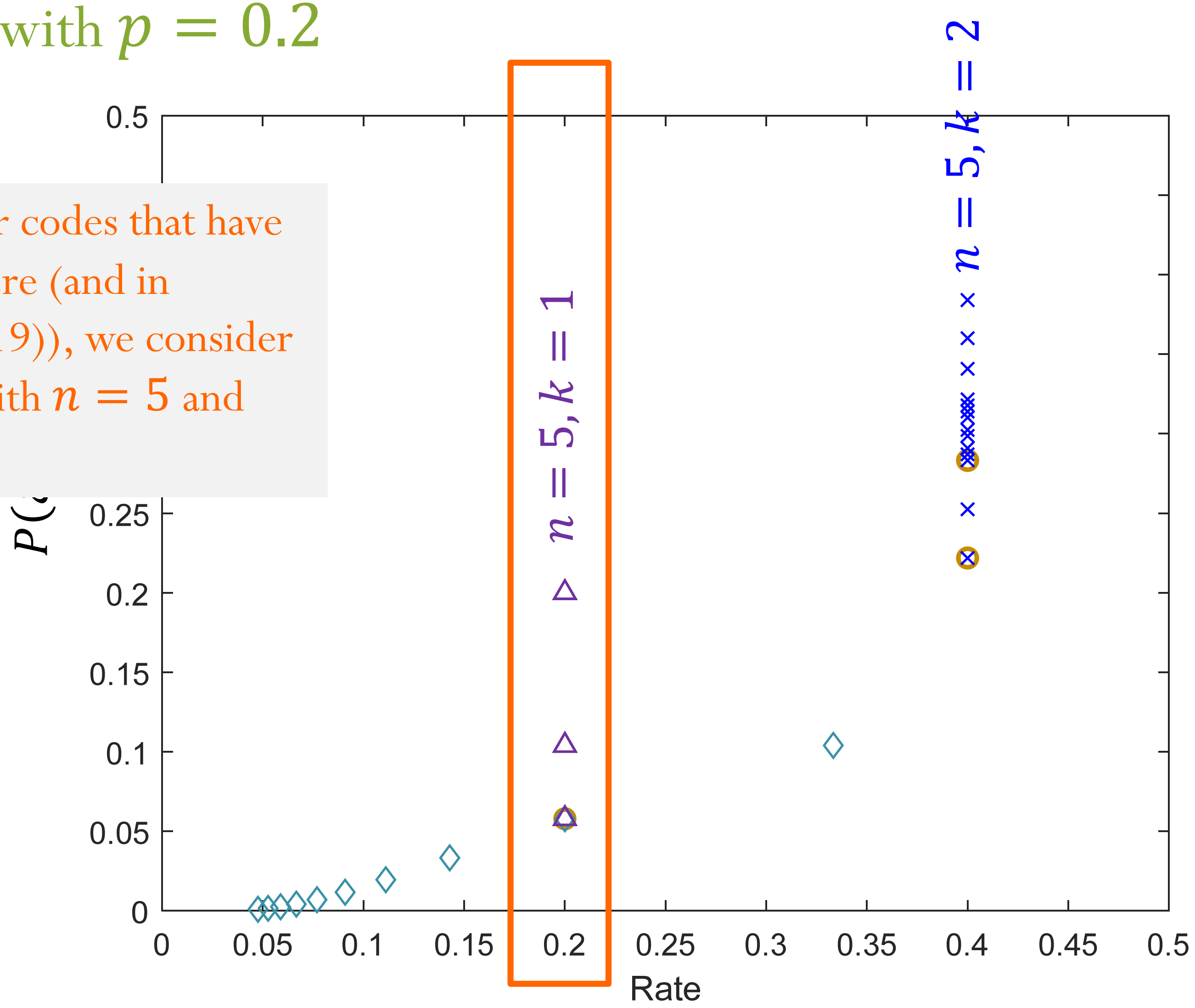
Achievable Performance

BSC with $p = 0.2$



Achievable Performance

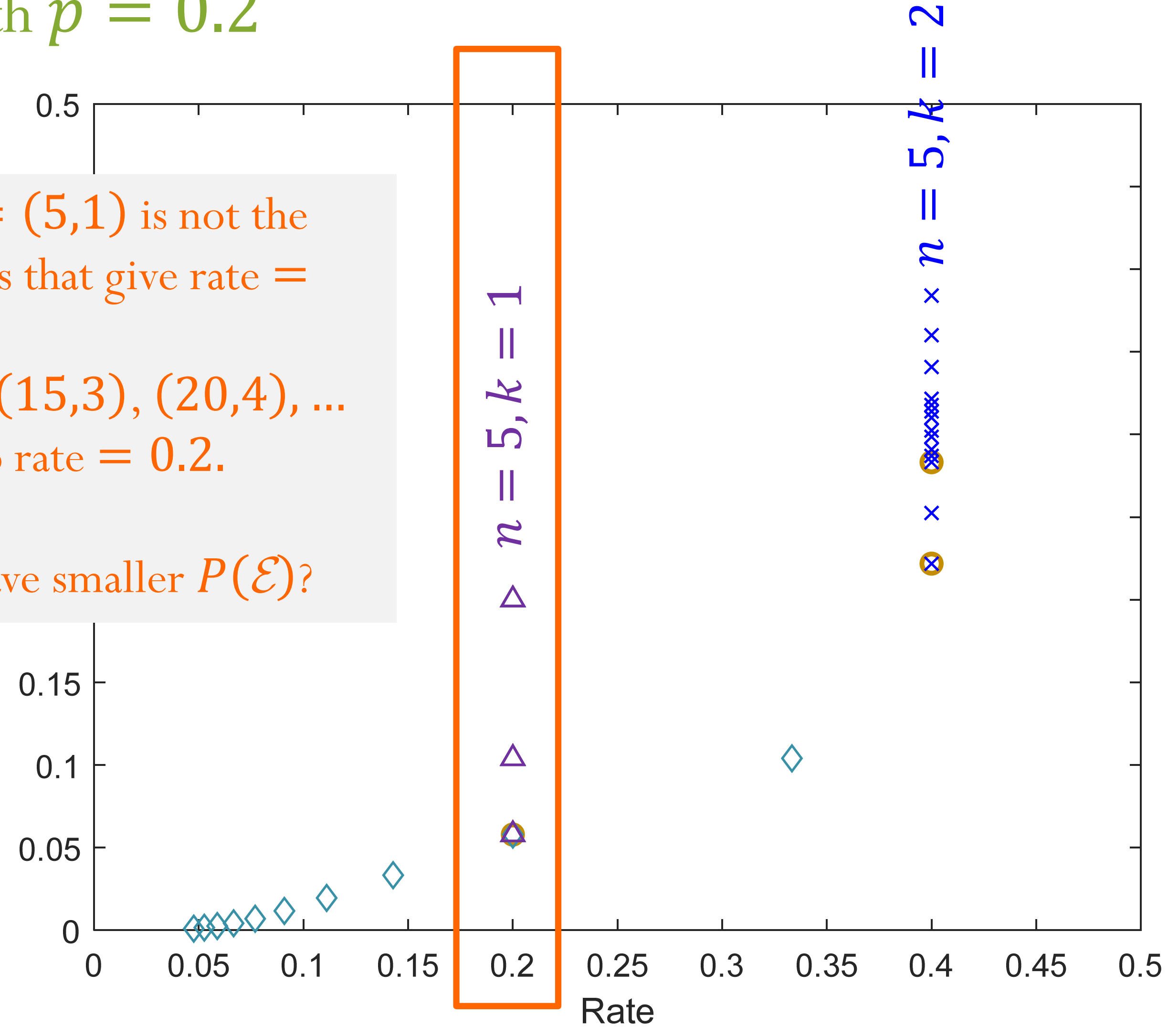
BSC with $p = 0.2$



There are other codes that have rate $= 0.2$. Here (and in Exercise 9 (2019)), we consider all the codes with $n = 5$ and $k = 1$.

Achievable Performance

BSC with $p = 0.2$

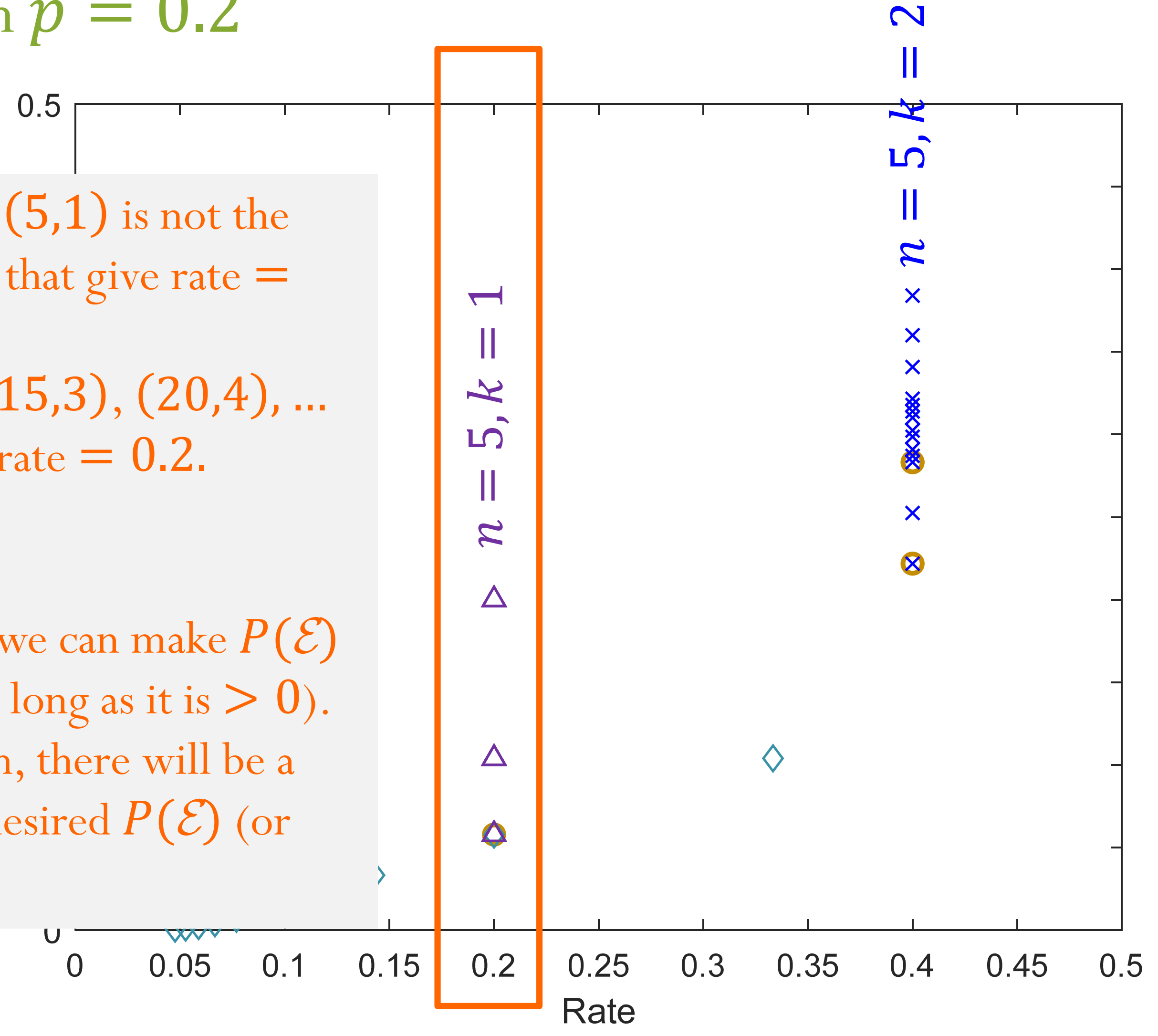


Note that $(n, k) = (5,1)$ is not the only family of codes that give rate $= 0.2$.
$(n, k) = (10,2), (15,3), (20,4), \ldots$ also corresponds to rate $= 0.2$.

Will these codes have smaller $P(\mathcal{E})$?

Achievable Performance

BSC with $p = 0.2$



Note that $(n, k) = (5,1)$ is not the only family of codes that give rate $= 0.2$.
$(n, k) = (10,2), (15,3), (20,4), \ldots$ also corresponds to rate $= 0.2$.

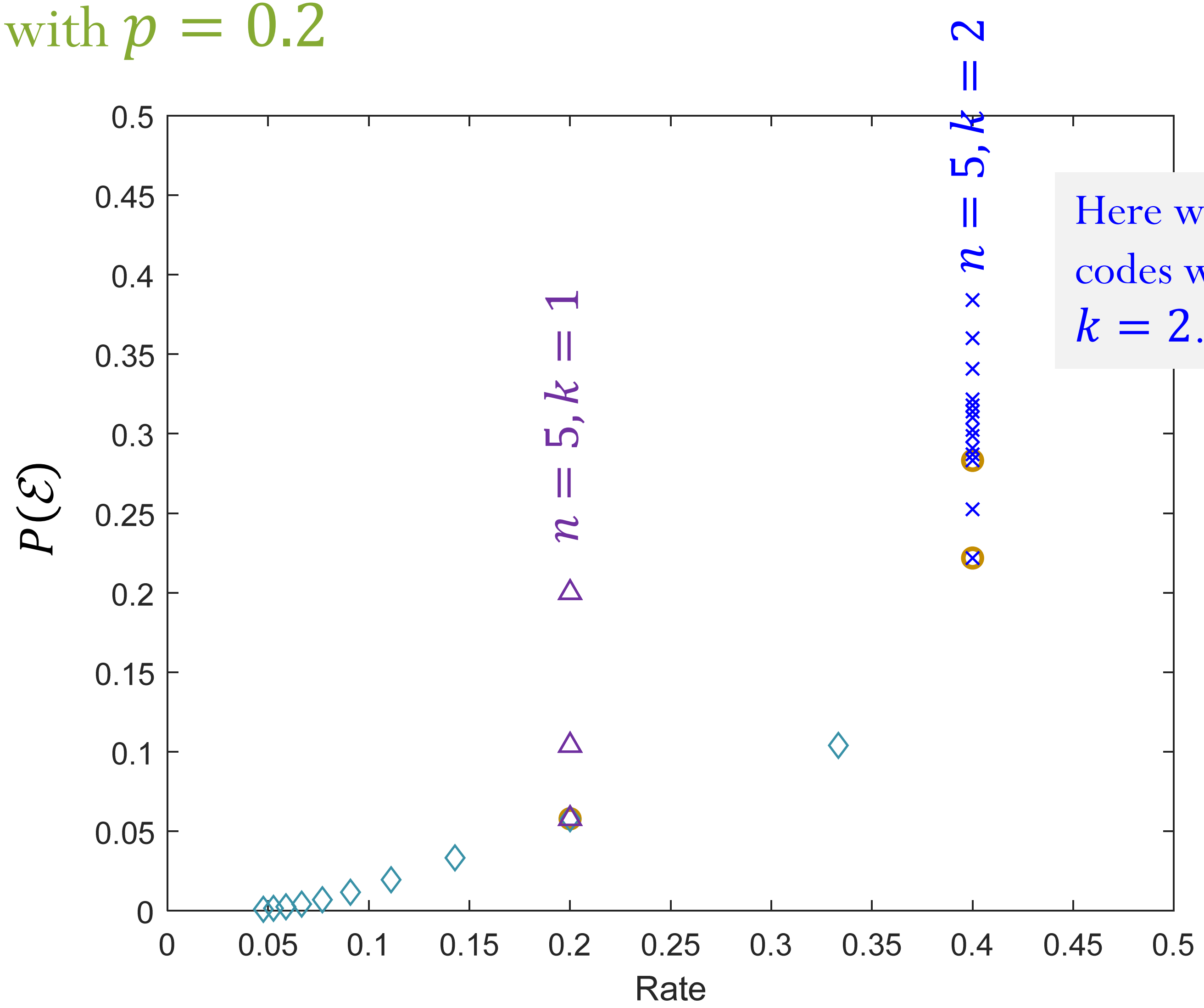
At rate $= 0.2$,
Shannon found that we can make $P(\mathcal{E})$ as small we want (as long as it is > 0).
With n large enough, there will be a code that gives the desired $P(\mathcal{E})$ (or smaller).

Reliable communication

- **Reliable communication** (at a particular rate) means arbitrarily small error probability can be achieved (at that rate).
- In our example, Shannon showed that reliable communication is achievable at rate $= 0.2$.
- Turn out that reliable communication is <u>not</u> achievable at rate $= 0.4$.

Achievable Performance

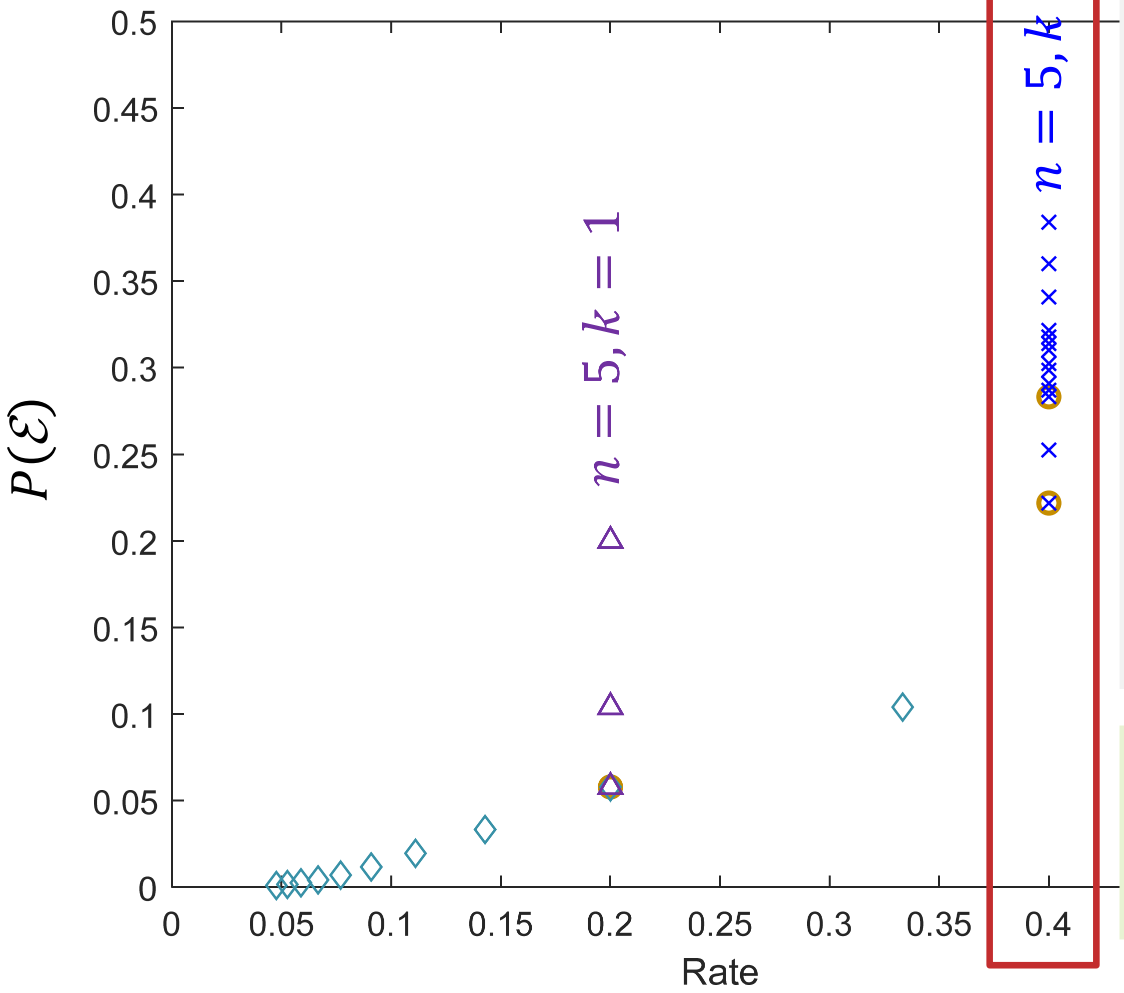
BSC with $p = 0.2$



Here we consider all the codes with $n = 5$ and $k = 2$.

Achievable Performance

BSC with $p = 0.2$



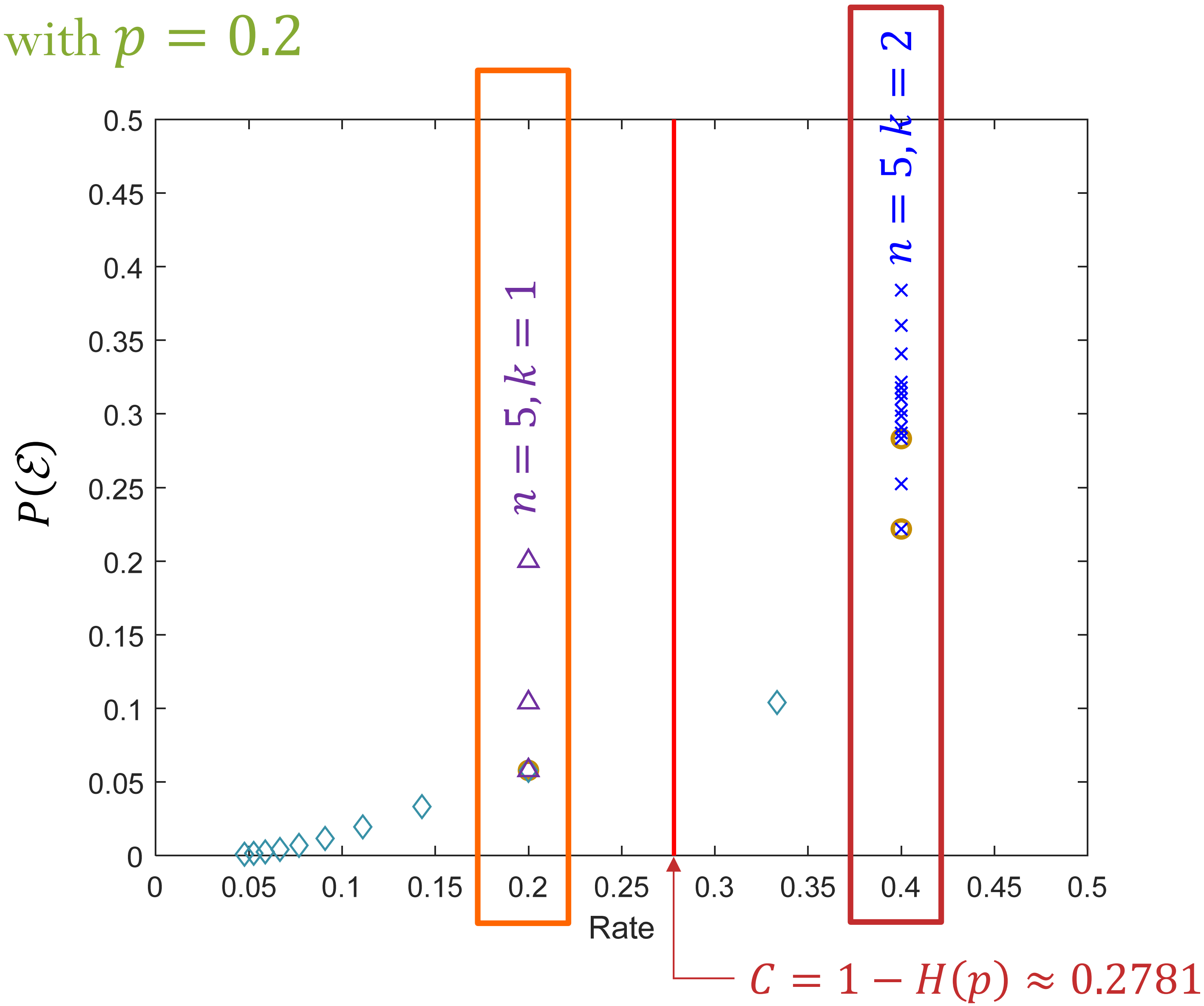
Note that $(n, k) = (5,2)$ is not the only family of codes that give rate $= 0.4$. $(n, k) = (10,4), (15,6), (20,8), \ldots$ also corresponds to rate $= 0.4$.

At rate $= 0.4$, Shannon found that we **cannot** make $P(\mathcal{E})$ as small as we want; even when we use large n.

So, how can we determine which rate can have arbitrarily small $P(\mathcal{E})$?

Achievable Performance

BSC with $p = 0.2$



Channel Capacity

Channel Capacity

[Section 4.2]

"**Operational**": max rate at which **reliable** communication is possible

Arbitrarily small error probability can be achieved.

"**Information**": $\max_{\underline{\mathbf{p}}} I(X;Y)$ [bpcu]

[Section 4.3]

Shannon [1948] showed that these two quantities are actually the same.